Quantum Hoare Type Theory

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EPiQC

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In classical programming

Hoare triples are used to reason about state changes.

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For quantum programming?

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For example, the type of the `alloc` primitive from HTT:

$$\forall\alpha.\Pi x : \alpha.\{\mathtt{emp}\}\ y : \mathtt{nat}\ \{y \mapsto_\alpha x\}$$

which is a polymorphic function that takes as input x of any type α and returns a new location y of type `nat` after initializing it with x.

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QIO is a monadic interface for quantum programming implemented in Haskell

QIO monad is indexed by the type of computational result.

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mkQbit :: Bool → QIO Qbit        -- initialization
applyU :: U → QIO ()             -- apply a unitary
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U is monoid with sequencing as its operation
and identity as the neutral element.

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But how do we reason about these programs?

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Successful type checking implies correctness
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Many exciting challenges ahead!